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Acoustic beaming and reflexion from wave-bearing surfaces

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The field radiated by an acoustic monopole in the presence of an infinite membrane, or plate, is studied, with emphasis on the case when fluid loading effects are small and when a free wave in the surface has supersonic phase speed relative to the fluid. Coupling between fluid and surface is then specified by a Mach angle θ_M and by a fluid loading parameter ϵ, with $\epsilon \ll 1$. Asymptotic expressions for the field are derived which are uniform in the observation angle θ, measured from the surface. Previous descriptions have suggested the formation of a strong two-dimensional beaming effect along the surface of the Mach cone $\theta = \theta_M$. Here it is shown that this effect is a spurious consequence of non-uniform asymptotics. A beam is indeed formed, and persists without attenuation or distortion to large distances $k_0R \sim \epsilon^{-2}$. However, the beam amplitude is small compared with that of the three-dimensional reflected field, while at distances $k_0R \gg \epsilon^{-2}$ only the reflected wave survives. Some interesting features of the reflexion coefficient and of the field near to the membrane are also discussed. In particular, it is shown that the pressure field generated by a subsonic surface wave is also confined to a conical zone, the transition across the generators of the cone being described by Fresnel functions of a familiar kind.

1. Introduction

Problems involving the coupled motion of a wave-bearing surface and a compressible fluid arise frequently. In some cases it is necessary to account for the modification of the field radiated by acoustic sources due to the induced surface waves, while in others the influence of fluid loading on the parameters describing the surface response is required. We shall consider here the first aspect of the problem in the very simplest case, that in which the surface is formed by an infinite homogeneous membrane. The essential features are demonstrated most clearly in this context, and the extension to the more realistic case of a thin elastic plate is easily made. Standing-wave modes in the surface are of course excluded by the infinite model, but any attempt to include effects due to finiteness or inhomogeneity of the surface appears premature until a proper description of the simplest case has been achieved.

Problems of the kind considered here are in no way novel, and have been treated in some detail in the recent book by Morse & Ingard (1968). Analysis is given there of the motion of a coupled fluid-plate system when the plate is either driven by a line force, or irradiated by a point source or plane wave acoustic field. However, the mathematics used is invalid in the region of the field where the most interesting phenomena occur, significance being attached, in particular, to an acoustic beaming generated by a supersonic surface wave, although the analysis cannot deal with this effect. Moreover, the beam described by Morse & Ingard propagates a small, though finite, amount of energy to infinity. This contradicts the reflexion principle of Ffowcs Williams (1965), that the acoustic power delivered by a source cannot be more than doubled by the presence of an infinite plane homogeneous surface. The persistence of the beam to infinity and the apparent power excess merely reflect a non-uniformity in the asymptotic methods used, as we shall see later.

The present problem appears to have been first discussed by Lamb (1957). His treatment does not differ essentially from that of Morse & Ingard, and is, in fact, less accurate in that it does not include all possible free modes of the coupled system. Feit (1966) considers a similar problem, that of a thick elastic plate driven by a point force and immersed in fluid. His argument for the relevance of this problem is that, when frequencies are sufficiently high that supersonic waves can propagate in a plate, the usual assumptions leading to the thin-plate equation cannot be made. There is no reason to doubt the validity of this argument, but Feit's work does not seem to settle the general problem, as he was concerned mainly with aspects of the reflected field. The structure of the pressure fields generated by subsonic and supersonic plate waves was not dealt with in that work. The radiation from a thick elastic plate under line-force excitation is also the subject of a long paper by Lowenthal (1964). Lowenthal gives an extensive discussion of supersonic surface waves (which, in the electromagnetic analogue, Marcuvitz (1956) has termed 'leaky waves'), but does not show how the conical beam which they generate must ultimately decay. Not surprisingly, his statements about the types of free mode which will feature in the solution as the values of plate and fluid parameters are varied are incomplete, for in the case of the elastic plate these modes are determined by the roots of a quintic. Here we adopt a much less ambitious approach and consider the simpler case of a membrane for which the free modes are determined by the roots of a cubic (equation (3.2)), allowing a complete description of the nature of these modes.

The beaming effect mentioned earlier might be of great practical importance, if it could be realized, and warrants a complete re-examination of the problem with a view to determining the precise form of the field along the beam direction. We shall see that a beam can indeed be formed, and that it will persist over a great distance if fluid loading effects are small. At still greater distances, however, the beam spreads, and we are left simply with the reflected field. Provided the source lies closer than a wavelength to the membrane, the total field in the beaming direction then vanishes at sufficiently great distances since the reflexion coefficient for that direction is equal to -1. Thus the two-dimensional beaming along a conical surface disappears with increasing distance and in the end the cone becomes a null surface of the total field.

The paper ends with a discussion of properties of the reflexion coefficient and of the field generated by subsonic surface waves. A feaure which seems not to have been previously noticed in the acoustic context is that the pressure field so induced is confined more or less abruptly beneath the surface of a certain cone. This cone acts as a shadow boundary for subsonic waves in the same manner as the Mach cone for supersonic waves.

2. Formulation

An infinite membrane lies in the plane $x_3 = 0$, with fluid on one side, $x_3 > 0$ say. A time factor $\exp(-i\omega t)$ is assumed, with $\omega > 0$, and the coupled motion of membrane and fluid is generated by a monopole source at $(0, 0, h)$. The acoustic wave-number ω/c_0 is denoted by k_0, while k_m is the free wave-number in the membrane in a vacuum. The sound speed in the fluid is c_0, and $c_m = (T/m)^{\frac{1}{2}}$ is the wave speed in the membrane, which has tension T and specific mass m.

The incident pressure field will be taken as

$$p_0(x, x_3) = \frac{\exp(ik_0 R_1)}{R_1}, \tag{2.1}$$

where
$$R_1^2 = x_1^2 + x_2^2 + (x_3 - h)^2 \equiv x^2 + (x_3 - h)^2.$$

A Hankel transform

$$\tilde{p}_0(s, x_3) = \int_0^\infty p_0(x, x_3)\, J_0(sx)\, x\, dx \tag{2.2}$$

may be evaluated from tables (Erdélyi *et al.* 1954), to give

$$\tilde{p}_0(s, x_3) = \gamma^{-1} \exp\{-\gamma|x_3 - h|\}. \tag{2.3}$$

Here, for real values of s, $\gamma(s)$ is defined by

$$\left.\begin{aligned} \gamma(s) &= (s^2 - k_0^2)^{\frac{1}{2}} \quad (k_0 \leqslant |s| < \infty), \\ &= -i\,(k_0^2 - s^2)^{\frac{1}{2}} \quad (0 \leqslant |s| \leqslant k_0), \end{aligned}\right\} \tag{2.4}$$

positive values of the radicals being implied. Subsequently we shall need to regard s as a complex variable, in which case $\gamma(s)$ is defined as that branch of $(s^2 - k_0^2)^{\frac{1}{2}}$ which reduces to (2.4) on the real axis, branch cuts being taken from $+k_0$ to infinity in the first quadrant and from $-k_0$ to infinity in the third.

Now let $p(x, x_3)$ denote the scattered pressure, $\tilde{p}(s, x_3)$ its Hankel transform. Then the Helmholtz equation

$$(\nabla^2 + k_0^2)\, p(x, x_3) = 0$$

and the condition that $\tilde{p}(s, x_3)$ shall either vanish, or represent an outgoing wave, as $x_3 \to +\infty$, are satisfied by

$$\tilde{p}(s, x_3) = \tilde{p}(s, 0) \exp(-\gamma x_3). \tag{2.5}$$

Next, let $y(x)$ be the membrane displacement in the positive x_3 direction. Then the kinematic condition on the membrane is

$$-\frac{\partial}{\partial x_3}(\tilde{p}+\tilde{p}_0)(s, 0) = -\rho\omega^2\tilde{y}(s),$$

and with (2.3) and (2.5) gives

$$\exp(-\gamma h)-\gamma\tilde{p}(s, 0) = \rho\omega^2\tilde{y}(s), \tag{2.6}$$

ρ being the fluid density. The dynamic condition

$$(T\nabla_1^2+m\omega^2)\,y(x) = p_0(x, 0)+p(x, 0)$$

in which ∇_1^2 denotes the surface Laplacian, gives

$$(-Ts^2+m\omega^2)\,\tilde{y}(s) = \gamma^{-1}\exp(-\gamma h)+\tilde{p}(s, 0). \tag{2.7}$$

Writing $\mu = \rho/m$, $k_m^2 = m\omega^2/T$, a particular solution for surface pressure is found by eliminating $\tilde{y}(s)$ from (2.6) and (2.7) to yield

$$\tilde{p}(s, 0) = \frac{\exp(-\gamma h)}{\gamma}\left[\frac{(s^2-k_m^2)\,\gamma+\mu k_m^2}{(s^2-k_m^2)\,\gamma-\mu k_m^2}\right], \tag{2.8}$$

from which the radiated pressure follows as

$$p(x, x_3) = \int_0^\infty \tilde{p}(s, 0)\,J_0(sx)\exp(-\gamma x_3)\,s\,ds. \tag{2.9}$$

Regarded as an integral in the complex s plane, the path of integration must avoid the branch cut in the first quadrant by means of an indentation below the point $s = k_0$. Similar indentations *below* any poles of $\tilde{p}(s, 0)$ on the positive real s axis will be found to yield a pressure field consisting of outgoing waves only as $|\mathbf{x}| \to \infty$. Then it will be neither necessary, nor possible, to add to (2.9) any regular solution of the homogeneous problem, for any such solution would contain standing waves violating the radiation condition.

We evaluate the integral (2.9) by first writing

$$2J_0(sx) = H_0^{(1)}(sx)+H_0^{(2)}(sx).$$

It is easy to show that there are no poles of the integral in the fourth quadrant, and the path for the integral involving $H_0^{(2)}(sx)$ may therefore be deformed directly onto the negative imaginary axis. Use of the identity $H_0^{(2)}(it) \equiv -H_0^{(1)}(-it)$, for real t, allows that integral to be expressed as one involving $H_0^{(1)}(sx)$ taken along the positive imaginary axis, with the result that

$$p(x, x_3) = \frac{1}{2}\int_C \tilde{P}(s, 0)\,H_0^{(1)}(sx)\exp(-\gamma x_3)\,s\,ds, \tag{2.10}$$

where the path C runs along the axes from $i\infty$ through 0 to $+\infty$, with indentations below singularities in $(0, \infty)$ as discussed above.

The simplest way of proceeding now is to use the elegant representation described in detail by Clemmow (1966), mapping the cut s plane on to a strip of the $\Theta = \alpha + i\beta$ plane according to

$$s = k_0 \cos\Theta \quad (0 \leqslant \alpha \leqslant \pi). \tag{2.11}$$

The value of γ in the strip is $-ik_0 \sin\Theta$, and figure 1 shows the transformed contour C'. It is possible to persevere with the s plane, and despite the increased algebraic complexity it is instructive to do so, though we defer discussion of this until §5.

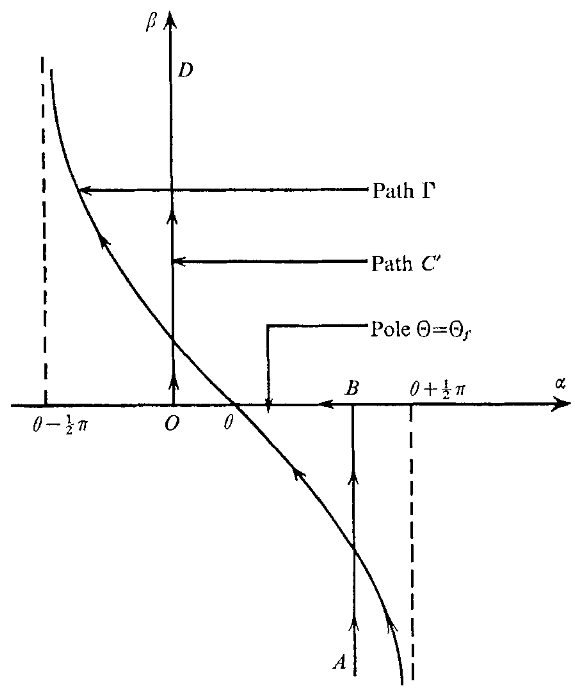

FIGURE 1. The $\Theta = \alpha + i\beta$ plane, showing the path C' ($ABOD$) of integration, and the path Γ of steepest descent. The path C' is to be regarded as indented to the right of any subsonic pole, such poles lying on the line OD. The points $ABOD$ correspond to the points $s = i\infty$, 0, k_0, and $+\infty$ respectively.

Let (R_2, θ) be polar co-ordinates based on the image point $(0, 0, -h)$ so that $x = R_2 \cos\theta$, $x_3 + h = R_2 \sin\theta$, $0 < \theta \leqslant \frac{1}{2}\pi$. Then we have

$$p(R_2, \theta) = -\tfrac{1}{2}ik_0 \int_{C'} \exp\left[ik_0 R_2 \cos(\Theta - \theta)\right] \cos\Theta F(\Theta)\, d\Theta,$$

where

$$F(\Theta) = \left[\frac{(\cos^2\Theta - k_m^2/k_0^2)\, i \sin\Theta - \mu k_m^2/k_0^3}{(\cos^2\Theta - k_m^2/k_0^2)\, i\sin\Theta + \mu k_m^2/k_0^3}\right]$$
$$\times H_0^{(1)}(k_0 R_2 \cos\Theta \cos\theta) \exp(-ik_0 R_2 \cos\Theta\cos\theta). \tag{2.12}$$

Now the saddle-point of the function $ik_0 \cos(\Theta-\theta)$ is at $\Theta = \theta$, and the lines of steepest ascent and descent through that point are given by

$$\cos(\alpha-\theta)\cosh\beta = 1. \tag{2.13}$$

The path Γ of steepest descent is that branch of (2.13) which behaves like $\alpha-\theta = -\beta$ near $\Theta = \theta$, and we may deform the path C' onto Γ, there being no contribution to the integral, from arcs at infinity linking C' to Γ. The deformation does require a knowledge of the poles of the integrand, and further development of the integral (2.12) is therefore now postponed until §4.

3. Free modes of the coupled system

The poles of the integrand of (2.12) are the values of Θ for which

$$(\cos^2\Theta - k_m^2/k_0^2)\, i\sin\Theta + \mu k_m^2/k_0^3 = 0, \tag{3.1}$$

and these values of Θ correspond in the s plane to the wave-numbers of the possible free oscillations of the coupled system with surface deflexion $\exp(isx_1)$, with a suitable choice of axes. Instead of solving (3.1) directly, we can solve for the values of $\gamma = -ik_0 \sin\Theta$, noting that an admissible γ must satisfy $\operatorname{Im}\gamma \leqslant 0$ everywhere in $0 \leqslant \alpha \leqslant \pi$, while $\operatorname{Re}\gamma \geqslant 0$ on $(\alpha = 0, \beta > 0)$ and on $(\alpha = \pi, \beta < 0)$. We have then

$$\gamma^3 + (k_0^2 - k_m^2)\gamma - \mu k_m^2 = 0, \tag{3.2}$$

an equation which can be solved by the usual methods. The explicit solutions are cumbersome and need not be quoted here, though they are useful in confirming the general arguments which follow.

Note that the roots of (3.2) have zero sum and positive product. Consequently, if (3.2) has three real roots, two must be negative and one positive, while if it has only one real root, that root must be positive and the two complex roots must have negative real parts. Which of these situations obtains is determined by whether the discriminant

$$\Delta = \mu^2 k_m^4 - \tfrac{4}{27}(k_m^2 - k_0^2)^3 \tag{3.3}$$

is negative or positive, respectively.

Suppose that $\Delta < 0$. Then the one real positive root for γ arises from values of Θ on $(\alpha = 0,\ \beta > 0)$ or $(\alpha = \pi,\ \beta < 0)$ and clearly, only the first of these is relevant, as the second can never be crossed in the deformation from C' to Γ. The two real negative values of γ are likewise irrelevant, for they arise from poles in the Θ plane lying on $(\alpha = \pi, \beta > 0)$, or $(\alpha = 0, \beta < 0)$. In this case then, there is only one pole of interest, $\Theta = \Theta_r$ say, with $\alpha_r = 0$, $\beta_r > 0$. In the s plane, this value of Θ corresponds to a pole $s = k_r$ on the positive real axis, with $k_r > k_0$. Indentation of the original contour C below k_r implies indentation of C' to the right of Θ_r, and therefore the pole at Θ_r may or may not be crossed in the deformation of C' on to Γ, depending upon the observation angle θ. This is a point to which we shall return in §5. We note that the free mode represented by this pole takes the form of a subsonic surface wave, travelling without attenuation over the membrane, and generating a near-field pressure wave confined to the layer $(k_r^2-k_0^2)^{\frac{1}{2}}x_3 \lesssim 1$.

Suppose next that $\Delta > 0$. Then there is one real positive solution of (3.2), and a complex conjugate pair with negative real part. The real positive solution for γ gives rise to a single relevant pole Θ_s say, with properties identical with those of Θ_r above. That one of the complex pair with $\operatorname{Im}\gamma > 0$ is inadmissible, is noted earlier in this section. This leaves us with the third root for γ, which arises from a pair of values of Θ, Θ_f and $\pi-\Theta_f$ say, and it is easy to see that $\operatorname{Re}\Theta_f < \frac{1}{2}\pi$, $\operatorname{Im}\Theta_f < 0$. The pole at $\pi-\Theta_f$ is therefore again irrelevant, but that at Θ_f may be captured in the deformation from C' to Γ depending upon the values of θ, μ/k_0 and k_m/k_0. Also, the pole at Θ_f may represent a free mode in which the phase velocity of surface waves is either subsonic or supersonic relative to the sound speed c_0—i.e. the image wave-number k_f in the s plane may have real part greater than or less than k_0, depending upon the ratios k_m/k_0, μ/k_0. However, it is not hard to see that any pole of the Θ_f kind which is crossed in the path deformation is *necessarily of the supersonic type.* The pressure field in the resulting free mode has the structure $\exp[ik_f x-\gamma(k_f)x_3]$, where we have

$$\operatorname{Re}k_f < k_0, \quad \operatorname{Im}k_f > 0, \quad \operatorname{Re}\gamma(k_f) < 0, \quad \operatorname{Im}\gamma(k_f) < 0.$$

The wave amplitude decays through radiation loss as the wave propagates supersonically along the x_1 axis, but increases exponentially with increase of x_3 at any fixed x_1 location. This is because one can regard a value of the pressure as generated at a point on the membrane and then propagated unmodified along the characteristic (Mach) direction $dx_3 = dx_1\tan\theta_M$ through that point, where $\theta_M = \cos^{-1}\{\operatorname{Re}k_f/k_0\}$. Consequently, increasing x_3 is equivalent to displacing the emission point backwards over the membrane in the direction of exponentially increasing surface deflexion. The corresponding waves of this type in electromagnetic propagation have been termed 'leaky waves' (Marcuvitz 1956).

Thus, in the case $\Delta > 0$ there are two possibilities. In both we have a pole of the Θ_r type, giving rise to a subsonic surface wave. In addition, a pole of the Θ_f type may be captured in deforming C' on to Γ and this pole will always give rise to a supersonic surface wave, but the condition under which the pole will be captured is too complicated to be worth setting down here.

Note that the condition for one or two poles which can possibly be captured is not simply that $k_0 < k_m$ or $k_0 > k_m$ respectively, as stated by Morse & Ingard (1968, p. 691). The condition on Δ reduces to these simpler conditions only when μ/k_0 and μ/k_m are small compared with unity, so that fluid loading effects are small. As a counter example to the conditions of Morse & Ingard, take any fixed ratio k_0/k_m, write $\nu = (\mu k_m^2)^{\frac{1}{3}}$, and let $\mu/k_0 \to \infty$. Then $\Delta > 0$, we have a pole of the Θ_s variety, and also one at

$$\Theta_f = \tfrac{1}{3}\pi - i\ln(2\nu/k_0),$$

which can *never* be captured in the deformation of C' on to Γ. Whether the ratio k_0/k_m is larger, or smaller, than unity is irrelevant; in both cases only the Θs pole makes any contribution.

Analytical approximations to the poles can easily be obtained for any *fixed* value of k_0/k_m when the fluid loading parameter $\epsilon = \mu/k_0$ is either small or large (the value of ϵ being essentially the ratio of the inertia of the fluid within an acoustic wavelength of the surface to the inertia of the membrane itself). Thus if $\epsilon \ll 1$ and $k_m > k_0$, then $\Delta < 0$, and the pole Θ_r and associated wave-number k_r are given by

$$\left.\begin{aligned} \Theta_r &= i\cosh^{-1}(k_r/k_0), \\ k_r &= k_m\left\{1+\frac{\mu}{2(k_m^2-k_0^2)^{\frac{1}{2}}}\right\}. \end{aligned}\right\} \tag{3.4}$$

If $\epsilon \gg 1$, then $\Delta > 0$ regardless of the value of k_0/k_m but the pole at Θ_f given above has already been seen to play no part. The other pole is of the subsonic kind and is given by

$$\left.\begin{aligned} \Theta_s &= i\cosh^{-1}(\nu/k_0), \\ k_s &= \nu = (\mu k_m^2)^{\frac{1}{3}}. \end{aligned}\right\} \tag{3.5}$$

Finally, consider the case $\epsilon \ll 1$, $k_0 > k_m$, and introduce a Mach angle through the relation $k_m = k_0 \cos\theta_M$. A pole of both the subsonic and supersonic kind now exists, and both are capable of making a contribution to the pressure field for certain ranges of the observation angle θ. The subsonic pole is given by

$$\left.\begin{aligned} \Theta_s &= i\cosh^{-1}(k_s/k_0), \\ k_s &= k_0(1+\tfrac{1}{2}\epsilon^2\cot^4\theta_M), \end{aligned}\right\} \tag{3.6}$$

while for the supersonic pole,

$$\left.\begin{aligned} \Theta_f &= \theta_M - \tfrac{1}{2}i\epsilon\cot\theta_M \operatorname{cosec}\theta_M, \\ k_f &= -ik_0\sin\theta_M - \tfrac{1}{2}\epsilon k_0\cot^2\theta_M. \end{aligned}\right\} \tag{3.7}$$

This case will be examined in detail in the next section, the analysis then being sufficiently simple to allow the general features to emerge very clearly. A point which should, however, be emphasized, is that the approximations given above are valid for any fixed value of k_0/k_m as $\epsilon \to 0$ or $\epsilon \to \infty$, and the approximations are not uniformly valid in the ratio k_0/k_m.

4. The radiated field

We return now to the evaluation of (2.12) taking $\epsilon \ll 1$ and $k_0 > k_m$. Further, we entirely ignore the subsonic pole Θ_s for the present. Then it follows from the linear form of (2.13) near $\Theta = \theta$ and the smallness of ϵ that the pole at Θ_f (see 3.7) will be crossed when C' is deformed on to Γ provided

$$\frac{\epsilon\cot\theta_M}{2\sin\theta_M} < \theta_M - \theta. \tag{4.1}$$

The residue contribution is easily found to $O(\epsilon)$, after expansion of the Hankel function for $k_0 R_2 \cos\theta_M \cos\theta \gg 1$, in the form

$$p_f(R_2, \theta) \sim -\left(\frac{2\pi i k_0}{R_2}\right)^{\frac{1}{2}} \epsilon \left(\frac{\cot\theta_M}{\sin\theta_M}\right)\left(\frac{\cos\theta_M}{\cos\theta}\right) H\left(\theta_M - \theta - \tfrac{1}{2}\epsilon\frac{\cot\theta_M}{\sin\theta_M}\right)$$
$$\times \exp\left[ik_0 R_2 \cos(\theta_M - \theta) - \tfrac{1}{2}\epsilon k_0 R_2 \left(\frac{\cot\theta_M}{\sin\theta_M}\right)\sin(\theta_M - \theta)\right]. \quad (4.2)$$

If the presence of ϵ in the Heaviside function is neglected (though it should not be), this term represents the beaming action along the surface of the Mach cone $\theta = \theta_M$, as described by Morse & Ingard (1968). On that cone, the pressure has the two-dimensional structure $R_2^{-\frac{1}{2}} \exp(ik_0 R_2)$ with no exponential decay, and the beam continues to infinity. However, we shall see that (4.2) is not the only contribution to a beam of this kind, and that the field (4.2) is annihilated at sufficiently large distances.

Considering now the integral along Γ, it can be shown that

$$|\cos\Theta|^2 > \tfrac{3}{4}(1 - \sin\theta)$$

for all Θ on Γ. If, therefore, we take $k_0 R_2 \cos\theta (1 - \sin\theta)^{\frac{1}{2}} \gg 1$, the argument of the Hankel function will be uniformly large on Γ, and the function may be replaced by its asymptotic form to give a field

$$p_\Gamma(R_2, \theta) \sim -\left(\frac{ik_0}{2\pi R_2 \cos\theta}\right)^{\frac{1}{2}} \int_\Gamma \exp[ik_0 R_2 \cos(\Theta - \theta)]$$
$$\times \cos^{\frac{1}{2}}\Theta \left[\frac{(\cos^2\Theta - \cos^2\theta_M)\, i \sin\Theta - \epsilon\cos^2\theta_M}{(\cos^2\Theta - \cos^2\theta_M)\, i \sin\Theta + \epsilon\cos^2\theta_M}\right] d\Theta. \quad (4.3)$$

The simple method of steepest descent does not give an asymptotic estimate of this integral which is uniformly valid for values of θ near θ_M. This fact was noted by Lamb (1957), though he did not make the appropriate correction, and it arises because the pole at Θ_f coalesces with the saddle-point as $\epsilon \to 0$. One method of ensuring a uniformly valid approximation is given by Jones (1964, p. 689). We have to evaluate an integral of the form

$$\int_\Gamma \exp[ik_0 R_2 \cos(\Theta - \theta)]\, G(\Theta)\, d\Theta,$$

where $G(\Theta)$ has a simple pole at Θ_f. We therefore isolate the singularity explicitly and approximate the remainder of $G(\Theta)$ by its value at the saddle-point. Then the usual approximation to the phase and path of integration leads to an integral of the form

$$\int_{-\infty}^{+\infty} \frac{\exp(-k_0 R_2 \beta^2)}{\theta - \Theta_f + (i-1)\beta}\, d\beta,$$

and this can be evaluated in terms of Fresnel or error functions, these providing the natural mathematical expression for the necessary smoothing of the discontinuous field $p_f(R_2, \theta)$.

Defining

$$\eta = \tfrac{1}{2}(i+1)\left(\theta_M - \theta - \tfrac{1}{2}i\epsilon \frac{\cot\theta_M}{\sin\theta_M}\right), \tag{4.4}$$

$$R_* = \left[\frac{(\cos^2\theta - \cos^2\theta_M)\, i\sin\theta - \epsilon\cos^2\theta_M}{(\cos^2\theta - \cos^2\theta_M)\, i\sin\theta + \epsilon\cos^2\theta_M}\right], \tag{4.5}$$

we find that if $\operatorname{Im}\eta > 0$ (i.e. if (4.1) is satisfied) then

$$p(R_2\, \theta) = p_f(R_2, \theta) + \exp[\tfrac{3}{4}\pi i]\,(\pi k_0/2R_2)^{\frac{1}{2}}\, R_*(\theta - \Theta_f)\exp[ik_0R_2 - \eta^2 k_0 R_2] \times \operatorname{erfc}[-i\eta(k_0R_2)^{\frac{1}{2}}], \tag{4.6}$$

while if the reverse of (4.1) holds, $\operatorname{Im}\eta < 0$, then

$$p(R_2, \theta) = -\exp[\tfrac{3}{4}\pi i]\,(\pi k_0/2R_2)^{\frac{1}{2}}\, R_*(\theta - \Theta_f)\exp[ik_0R_2 - \eta^2 k_0 R_2] \times \operatorname{erfc}[+i\eta(k_0R_2)^{\frac{1}{2}}]. \tag{4.7}$$

It is easy to see that the field defined by (4.6) and (4.7) is continuous across the surface $\operatorname{Im}\eta = 0$. Note that this surface is displaced slightly below the Mach cone $\theta = \theta_M$ because of the small, though finite, fluid loading. Expressions apparently differing from those above may be derived by either of the methods given by Clemmow (1966, p. 57) in which the pole is isolated in somewhat different ways, but all such expressions are asymptotically equivalent to those given here.

Suppose now that $|\eta|(k_0R_2)^{\frac{1}{2}} \gg 1$ (in addition to the previous requirements, which amount to $k_0R_2 \gg 1$ provided θ is not near to $\frac{1}{2}\pi$). Then we may use the asymptotic form

$$\operatorname{erfc} z \sim \pi^{-\frac{1}{2}} z^{-1}\exp[-z^2]$$

as $z \to \infty$, since the error functions in both (4.6) and (4.7) have positive real parts of their respective arguments. Above the cone $\operatorname{Im}\eta = 0$, i.e. for

$$\theta > \theta_M - \tfrac{1}{2}\epsilon\frac{\cot\theta_M}{\sin\theta_M} = \theta'_M \text{ say,}$$

p_f is zero and we have simply from (4.7)

$$p(R_2, \theta) \sim R_* \exp[ik_0R_2]/R_2.$$

This holds when either $\theta - \theta'_M = O(\epsilon)$ or smaller and $k_0R_2 \gg \epsilon^{-2}$, or when $\theta - \theta'_M = O(1)$ and $k_0R_2 \gg 1$.

Under the same conditions we have from (4.6)

$$p(R_2, \theta) \sim p_f(R_2, \theta) + R_*\exp[ik_0R_2]/R_2$$

for values of θ less than θ'_M. In the case when $\theta - \theta'_M = O(\epsilon)$ or smaller and $k_0R_2 \gg \epsilon^{-2}$, p_f is negligible. For the minimum possible value of $\sin(\theta_M - \theta)$ is $\sin(\frac{1}{2}\epsilon\cot\theta_M/\sin\theta_M)$ and therefore from (4.2) p_f is of order $\exp[-\epsilon^2 k_0 R_2]$. In the case when $\theta - \theta'_M = O(1)$, p_f is of order $\exp[-\epsilon k_0 R_2]$, and is negligible when $k_0R_2 \gg \epsilon^{-1}$.

In summary then we have

$$p(R_2, \theta) \sim R_* \exp[ik_0 R_2]/R_2 \tag{4.8}$$

for θ near to θ'_M and $k_0 R_2 \gg \epsilon^{-2}$, or for $\theta \gg \theta'_M$ and $k_0 R_2 \gg 1$, or for $\theta \ll \theta'_M$ and $k_0 R_2 \gg \epsilon^{-1}$. Equation (4.8) is just a statement of the reflexion principle (Ffowcs Williams 1965), R_* being the reflexion coefficient for a plane wave whose propagation vector makes an angle θ with the surface. The reflexion principle holds well above the Mach cone $\theta = \theta'_M$ at distances $k_0 R_2 \gg 1$, well below the Mach cone at greater distances $k_0 R_2 \gg \epsilon^{-1}$, and near the Mach cone provided the distance is very great, $k_0 R_2 \gg \epsilon^{-2}$. We note that $|R_*| = 1$ for all angles θ, and so the acoustic power radiated in any direction cannot be increased over the free field value by more than a factor of four through the presence of the membrane.

This leaves us with one region still to consider, that in which $|\theta - \theta'_M| = O(\epsilon)$ or smaller and $1 \ll k_0 R_2 \ll \epsilon^{-2}$. Here the field is given by (4.6) or (4.7) as

$$p(R_2, \theta) \sim (\pi i k_0/2R_2)^{\frac{1}{2}} \tfrac{1}{2}\epsilon (\cot\theta_M/\sin\theta_M) \exp[ik_0 R_2], \tag{4.9}$$

and takes the form of a conical beam with a two-dimensional structure. Note that (4.9) is not identical with (4.2), and that the residue contribution (4.2) given by Morse & Ingard (1968) fails to describe the beam correctly even where the beam does exist. Moreover, it is not true that the beam carries even a significant fraction of the energy. For the ratio of amplitude along the beam to that of the reflected field which dominates elsewhere is of order $\epsilon(k_0 R_2)^{\frac{1}{2}}$, and this is small throughout the range of R_2 for which the beam persists. Consequently, little emphasis should be placed on the extremely pronounced directivity pattern which would result if the beam were able to continue to infinity without distortion.

The features discussed above, of the field generated when circumstances allow propagation of supersonic elastic waves in the surface, are sharply defined when fluid loading effects are small. If those effects are not small, but conditions still such as to permit supersonic surface waves, the pole at Θ_f will still lie close to the path Γ of steepest descent for a certain value of θ but will not tend to coalesce with the saddle-point. The sharpness and persistence of the beaming effect will then not be as pronounced, but otherwise the conditions reached above will remain qualitatively valid and require no further discussion.

5. Subsonic surface waves

Suppose now that the only pole of the integral of (2.12) is one of the subsonic type, denoted by Θ_r or Θ_s in §3 and corresponding to a free mode with wave-number κ say. This pole will be captured in the deformation from C' to Γ if

$$\theta < \cos^{-1}(k_0/\kappa),$$

giving rise to a residue contribution

$$p_r(x, x_3) = \tfrac{1}{2}\pi i\, H_0^{(1)}(\kappa x) \exp[-(\kappa^2 - k_0^2)^{\frac{1}{2}}(x_3 + h)] \times \left[\frac{2\mu h_m^2 (\kappa^2 - k_0^2)^{\frac{1}{2}}}{2(\kappa^2 - k_0^2)^{\frac{3}{2}} + \mu h_m^2}\right] H\left(\cos^{-1}\frac{k_0}{\kappa} - \theta\right). \tag{5.1}$$

The steepest descent integral can be evaluated asymptotically as before, and the results are identical with those of (4.6) and (4.7), with (5.1) replacing the expression (4.2) for p_f and with η now defined as $\frac{1}{2}[i(\lambda-\theta)-(\lambda+\theta)]$ where $\kappa = k_0 \cosh \lambda$. The argument of the error functions is now uniformly large for all θ not near to $\frac{1}{2}\pi$ provided $k_0 R_2 \gg 1$, and the error functions may be replaced by their asymptotic forms to give

$$p \sim p_r + R_* \exp [ik_0 R_2]/R_2. \tag{5.2}$$

R_* is defined formally in (4.7), but is written more naturally in the form

$$R_* = \left[\frac{(Tk_0^2 \cos^2\theta - m\omega^2) + i\rho c_0 \omega \operatorname{cosec}\theta}{(Tk_0^2 \cos^2\theta - m\omega^2) - i\rho c_0 \omega \operatorname{cosec}\theta}\right], \tag{5.3}$$

which allows straightforward generalization.

In this case of subsonic surface wave, the rather curious angular cut-off is maintained to arbitrarily large distances. This effect is well-known in electromagnetic propagation (Clemmow 1966, p. 117) but appears to have been overlooked in the acoustic case treated by Lamb (1957) and Morse & Ingard (1968). Since Morse & Ingard work entirely in the s plane it is illuminating to resolve the discrepancy between their result and equations (5.1) and (5.2).

Consider again equation (2.10). Morse & Ingard take branch cuts from k_0 to $k_0 + i\infty$ and from $-k_0$ to $-k_0 - i\infty$, in which case $\operatorname{Re}\gamma > 0$ everywhere at infinity and the contour C can be deformed on to the edges of the branch cut from k_0 with capture of all poles in the first quadrant. The branch cut is then deformed to pass through the saddle-point $s = k_0 \cos\theta$ at an angle $\frac{3}{4}\pi$ with the positive σ axis ($s = \sigma + i\tau$), so that it follows the path of steepest descent Λ in the vicinity of the saddle-point. Such a deformation clearly cannot affect any *subsonic* poles, which all lie to the right of k_0. Of course, supersonic poles may be affected thereby, and it is a straightforward matter to recover the results of §4 by this method. However, it is essential that the path C be deformed on to the whole of Λ, and not just an approximation to Λ, in order for the correct structure of the surface wave field to be obtained. The equation of Λ can be found as

$$(\sigma - k_0 \cos\theta)(\sigma \cos\theta - k_0) = \tau \sin\theta \, (\sigma^2 - 2k_0 \sigma \cos\theta + k_0^2)^{\frac{1}{2}}, \tag{5.4}$$

and shows that Λ cuts the σ axis at $k_0 \cos\theta$ *and* at $\sigma = k_0 \sec\theta$, going to infinity with asymptotes $\tau = |\sigma| \cot\theta$. Therefore, a subsonic pole $s = \kappa$ will be captured in the deformation from C to Λ if $k_0 \sec\theta < \kappa$, and not otherwise, thus recovering (5.1).

Admittedly, the effect under discussion here may be of no more than academic interest. For if $\kappa \gg k_0$, the resulting surface wave pressure field is attenuated so rapidly with distance x_3 from the surface that the abrupt conical cut-off is probably inconsequential, while in our particular problem the amplitude of the wave corresponding to the marginally subsonic pole $s = k_s$ (equation (3.6)) is $O(\epsilon)$ so that the wave carries virtually no energy. A point to note in this connexion is that the energy carried by a subsonic surface wave decreases rapidly with increase of source height h from the surface, through the factor

$$\exp[-2h(\kappa^2 - k_0^2)^{\frac{1}{2}}]$$

which arises when the squared amplitude of (5.1) is taken.

We conclude with a brief discussion of some properties of the reflexion coefficient which are of great practical significance. The work of §4 does not require the source to be closer than a wavelength to the membrane. If, however, we do have the common circumstance that $hk_0 \ll 1$, it follows that the direct field p_0 from the source will be cancelled by the reflected field in the direction $\theta = \theta_M$, where the reflexion coefficient $R_* = -1$. Thus far from observing a sharp maximum in the radiated signal at great distances along the Mach cone, we should find instead a sharp drop to zero in the total signal, at least to order R_2^{-1}. This property holds also for emission in directions sufficiently close to the surface of the membrane, and the conclusion then is independent of the nature of the surface, provided only that it is homogeneous and has finite impedance (as all practical structures necessarily do have). When such a surface is excited by any source distribution within a wavelength of the surface, the total radiated field of the $R^{-1}\exp[ik_0R]$ type must vanish for those small values of θ which make $R_* \approx -1$. A genuine radiative field cannot be propagated along any homogeneous structure with finite impedance. The assumption commonly made in sonar design, for example, of infinite surface impedance, may be adequate for most values of θ, but should be replaced by the assumption that the surface is one of pressure-release for small values of θ. 'Small' here must not be taken too literally; a simple calculation shows that values of θ at least as large as 30° are still small in this sense in some typical underwater contexts. All one has to do to see this is to examine the range of angles sufficiently close to the surface that $R_* = -1$. Perhaps the most systematic method is to note that R_* is of the form $(a+ib)/(a-ib)$, and therefore that the power radiated in direction θ is obtained from the free-field value simply by multiplying by $4a^2/(a^2+b^2)$. The variation of this factor with θ shows most directly the way in which surface compliance reduces the power radiated in any direction. For the important case of a thin elastic plate with bending stiffness B, the functions a, b above are obtained from direct analogy with (5.3) as

$$a = Bk_0^4\cos^4\theta - m\omega^2,$$
$$b = \rho c_0\omega\,\mathrm{cosec}\,\theta,$$

with corresponding generalizations for more complicated equations describing the surface response.

If further details of the field for small angles are required, the next term of the steepest descent series must be found, though in general there will be no need to modify that series to allow for the presence of a pole. It is a straightforward matter to prove that

$$\int_\Gamma \exp[ik_0R_2\cos(\Theta-\theta)]\,G(\Theta)\,d\Theta = -\left(\frac{2\pi}{k_0R_2}\right)^{\frac{1}{2}}\exp[ik_0R_2-\tfrac{1}{4}i\pi]\,G(\Theta)$$
$$\times\left\{1-\frac{i}{2k_0R_2}\left(\frac{G''(\theta)}{G(\theta)}+\frac{1}{4}\right)+O(R_2^{-2})\right\}, \quad (5.5)$$

and this may be used in (4.3) to obtain a cumbersome expression of the form $R_2^{-2}\exp[ik_0R_2]$ for use near the membrane, or at great distances along the Mach cone. We might note that the electromagnetic analogue of this behaviour is well known (e.g. Tyras 1969, p. 148). Horizontal radio propagation over a finitely-conducting earth suffers from the fact that the reflexion coefficient is equal to -1, but the acoustic parallel does not seem to be widely appreciated.

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REFERENCES

CLEMMOW, P. C. 1966 *The Plane Wave Spectrum Representation of Electromagnetic Fields.* Pergamon.

ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F. G. 1954 *Tables of Integral Transforms*, vol. 2. Bateman Manuscript Project. McGraw-Hill.

FEIT, D. 1966 Pressure radiated by a point-excited elastic plate. *J. Acoust. Soc. Am.* **40**, 1489.

FFOWCS WILLIAMS, J. E. 1965 Sound radiation from turbulent boundary layers formed on compliant surfaces. *J. Fluid Mech.* **22**, 347.

JONES, D. S. 1964 *The Theory of Electromagnetism.* Pergamon.

LAMB, G. L. 1957 On the transmission of a spherical sound wave through a stretched membrane. *J. Acoust. Soc. Am.* **29**, 1091.

LOWENTHAL, S. 1964 Étude théoretique et expérimentale de l'interaction entre une plaque élastique plane et son rayonnement acoustique. *Annales de Radioélectricté,* **19**, 183.

MARCUVITZ, N. 1956 On field representations in terms of leaky modes or eigenmodes. *I.R.E. Trans. AP* **4**, 192.

MORSE P. M. & INGARD, K. U. 1968 *Theoretical acoustics.* McGraw-Hill.

TYRAS, G. 1969 *Radiation and Propagation of Electromagnetic Waves.* Academic.